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the outer segment the conversion of light into nerve-force may take place.

4. The outer granules being the nuclei of the inner cone- and rod-segments, probably maintain the integrity of these as living tissues, and are not directly concerned in their specific functions as organs of perception.

5. The primitive bacillary fibres are the link by which the cones and rods communicate through the inner granules and ganglion-cells with the optic nerve-fibres.

6. The smaller inner granules are nuclei of the oblique bacillary fibres in the inner granule-layer; or they may be small bipolar ganglion-cells, and act specifically on the forces transmitted through the oblique fibres from the cones and rods. The larger inner granules not being distinguishable by any definite structural characters from the smaller cells of the ganglionic layer, may agree with these latter cells in function.

7. Since the ganglion-cells (of the ganglionic layer) are fewer than the inner granules, and much fewer than the cones and rods, and since it is probable that these latter communicate with the optic nerve-fibres only through the ganglion-cells, it follows that one ganglion-cell probably is in correspondence with more than one inner granule and with several cones and rods. From this it is not an improbable conjecture that the cones and rods are disposed in groups, each of which is represented by one or more ganglion-cells the function of which is to connect or coordinate the individual action of the separate bacillary elements in their groups in a manner analogous to that attributed to the ganglion-cells of the spinal cord by Van der Kolk*.

8. There is a close general resemblance between the human fovea and that of the chameleon†.

II. Second Memoir "On Plane Stigmatics." By ALEXANDER J. ELLIS, F.R.S., F.C.P.S. Received May 26, 1866.

(Abstract.)

Let there be two groups of points upon a plane, termed, for distinction, *indices* and *stigmata* respectively, bearing such relations to each other that any one index determines the position of n stigmata, and any one stigma determines the position of m indices. The theory of these relations between indices and stigmata constitutes *plane stigmatics*. Each related pair of index X and stigma Y constitutes a *stigmatic point*, henceforth written "the s. point (xy).” The straight lines joining any index with each of its corresponding stigmata are termed *ordinates*. If, when

* I have an impression that I have seen this in a German author, but have not been able to find the passage again.

† H. Müller, "Ueber das Auge des Chamäleon," Wurzb. naturw. Zeitschr. Bd. iii. S. 36.

the index moves upon a straight line, the ordinate remains parallel to some other straight line, the relation between index and stigma is that expressed by the relation between abscissa and ordinate in the coordinate geometry of Descartes. When only one index corresponds to one stigma and conversely, and both indices and stigmata lie always on one and the same straight line, or the indices upon one and the stigmata upon another, the relations between indices and stigmata are those between homologous points in the homographic geometry of Chasles.

The general expression of the *stigmatic relation* is obtained by a generalization of Chasles's fundamental lemma in his theory of characteristics (*Comptes Rendus*, June 27, 1864, vol. lviii. p. 1175), clinants being substituted for scalars*. It results that in certain forms of the *law of coordination*, which "coordinates" the stigmata with the indices, there may be *solitary indices* which have no corresponding stigmata, and *solitary stigmata* which have no corresponding indices, and also *double points* in which the index coincides with its stigma (76)†. The particular case in which one index corresponds to one stigma and conversely, and no solitary index or stigma occurs, is termed a *stigmatic line* (henceforth written "s. line"), because the Cartesian case is that of a Cartesian straight line in ordinary coordinate geometry, but in the general s. line the figures described by index and stigma may be any directly similar plane figures (77). The investigation of this particular case occupies almost the whole of the *Introductory Memoir*. When one index corresponds to one stigma and conversely, but there is one solitary index and one solitary stigma, we have s. *homography*, provided the solitary index is distinct from the solitary stigma (79), and s. *involution* when the solitary index coincides with the solitary stigma (78), so called because they generalize the relations treated of under these names by Chasles.

When the relations between index and stigma are expressed by an equation of the second order, the s. curves are termed s. *conics*, because they include Cartesian conics as a particular case (80). Generally two stigmata correspond to each index, and two indices to each stigma, and there is no solitary index or stigma.

Two s. curves *intersect* when they have a common ordinate, and there-

* If OI be any fixed axis of reference, and AB any line upon a fixed plane passing through OI, we may consider that a line equal in magnitude and direction to AB can be formed from OI by the operation of turning OI through the $\angle$ (OI, AB), and altering its length in the ratio of the length of OI to that of AB. When all such lines AB lie upon the same plane, this operation is termed a *clinant* and represented by *ab* to distinguish it from the line AB. If the lines AB lay on different planes, the operation would be in general a *quaternion*. But if AB is parallel to OI, whatever be the plane on which it lies, the operation *ab* is termed a scalar. Clinants and scalars obey the laws of ordinary commutative algebra. Quaternions do not.

† The figures in parentheses refer to the articles of this "Second Memoir," which are numbered consecutively to those in the *Introductory Memoir* (Proceedings, April 6, 1865, vol. xiv. p. 176).

fore a common s. point. They *touch* when a slight alteration in the constants of one equation will make one of their common s. points into several, having their stigmata very near. When they have no common ordinate or s. point, they are *parallel* or *asymptotic*, according as the distances between the stigmata corresponding to a common index remain unchanged or, under certain circumstances, diminish infinitely (81).

It has been shown in the *Introductory Memoir* that a s. line can be determined by two stigmata corresponding to known indices, or by two points called the *direction-point* and *original stigma**. Taking either the two first-named or the two last-named points as the index and stigma in a stigmatic relation, we are able to make its expression denote a *linear relation*, which leads to the conception of enveloped s. curves, and therefore of *classes* of s. curves (82, 83).

In the above relations two ordinary geometrical points ("g. points") in a plane form the index and stigma. If we now take two s. points (that is, two relations of index to stigma), terming one *primary* and the other *secondary*, we are able, by means of two equations, to express a relation between these s. points, so that one primary shall correspond to n secondaries, and one secondary to m primaries. This is the *bipunctual relation* (84), and includes the whole subject of related s. curves, of which related Cartesian curves form a particular case. The most important and elementary cases are, first, those in which the related s. points lie respectively on known s. lines, s. involutions, s. homographies, or s. circles, and there is a stigmatic relation between the stigmata considered as forming two groups, one of indices and one of stigmata; and secondly, those in which one primary s. point corresponds to one secondary and conversely. In the latter case the relations between one index and stigma may be inferred from known relations between another index and stigma, or from the same index and a new stigma, or, as is most usual, from the same stigma and a new index. Hence follows the general theory of change of coordination (85). Homographic and homologic s. figures (86) are another example, in which one primary corresponds to one secondary s. point and conversely; but there is generally one *solitary s. line* in each s. figure such that no s. point taken upon it will have any corresponding s. point in the other figure.

Corresponding to the bipunctual relation there is a *bilinear relation* (87) in which one primary s. line corresponds to several secondary s. lines, and one secondary to several primaries, by means of the elements chosen to represent linear relations. To this belongs the *biradial relation* (87 b), where two pencils of s. rays, being drawn from known s. points, are determined generally by s. points having a bipunctual relation, and exceptionally

* The original stigma is the stigma of which the origin is the index. If through the s. point (ii) (that is, a s. point of which both index and stigma are the point I at the further extremity of the axis of reference OI) a s. line be drawn s. parallel to any given s. line, the original stigma of this parallel s. line is the direction-point of the given s. line. Direction-points are of great use in explaining "imaginary" lines and "imaginary" angles.

by their direction-points only, so that the relation between the rays can be expressed by a stigmatic relation between the direction-points considered as indices and stigmata. When this s. relation between the direction-points is a s. homography, we have pencils of *homographic s. rays*. The *reciprocal relation*, in which a s. point corresponds to several s. lines, and a s. line to several s. points, follows immediately (88).

Finally, the *multindicial relation* (89) shows how several indices may be made to correspond as one group to a single stigma or a group of stigmata, and thus leads to the geometrical expression of the algebraical theories of several independent variables. The means by which a theory of *solid stigmatics*, where the indices and stigmata are taken anywhere in space, may be expressed by quaternion equations is briefly pointed out (89 e).

The systematic explanation and precise expression for all these relations, together with some convenient notations relating especially to s. angles * (90), and to anharmonic ratios (91), occupy the first part of the present memoir. The remaining two divisions are devoted to a somewhat fuller consideration of s. homography (including s. involution) and s. conics.

It results (92) from the conception of *s. involution*, already given, that if S be the coincident solitary index and solitary stigma, and X, Y a corresponding pair of index and stigma, that is, if (xy) is a s. point on the s. involution, there will be two points, E, F (called *double stigmata*), in the same straight line with S, in which index and stigma coincide, forming the *double points* (ee), (ff), and then $se^2 = sf^2 = sx \cdot sy$, which implies that $\angle XSE = \angle ESY$, and $\angle XSF = \angle FSY$, and also that each of the lengths of SE, SF is a mean proportional between the lengths of SX and SY. It is readily shown that the two cases of involution of points in a line, usually acknowledged, are but the particular cases of XY lying on the line EF, or on a line through S perpendicular to EF. In the former case the double points lying on the line containing XY were readily found, and hence termed "real;" in the latter, as the method of investigation did not suffice to determine the double points, they were termed "imaginary." If A, B, C, D be any four indices, and A', B', C', D' their corresponding stigmata in a s. involution, then

$$\frac{ab \cdot cd}{ad \cdot cb} = \frac{a'b' \cdot c'd'}{a'd' \cdot c'b'}, \text{ and } \frac{ab \cdot cc'}{ac' \cdot cb} = \frac{a'b' \cdot c'c'}{a'c' \cdot c'b'};$$

that is, *the anharmonic ratio† of any four indices is the same as that of*

* If A, B be the direction-points of two s. lines, and O the origin, then $\frac{oa - ob}{1 - oa \cdot ob}$ is defined to be the s. tangent of the s. angle between these s. lines, and written $\tan aib$.

† The equality of *anharmonic* ratios is here taken to imply, not only the same relation of magnitude which would be included in the term if the four points A, B, C, D lay upon one straight line, and their corresponding points A', B', C', D' upon another straight line, but also an angular relation which is expressed by the notation of clinants, and which in the first of these equations is $\angle BAD + \angle DCB = \angle B'A'D' + \angle D'C'B'$. It is readily seen that this relation reduces to the usual relation of direction when the points lie on straight lines.

their corresponding four stigmata, and the anharmonic ratio of any three indices and stigma is the same as that of the corresponding three stigmata and index in a s. involution. If the index move on the characteristic circle whose centre is S and radius SE, the corresponding stigma moves in the opposite segment of the same circle, and in an opposite direction of rotation. Generally if the index move on any circle passing through E, F, the stigma moves on the segment of the same circle lying on the other side of EF, and in the opposite direction of rotation. If the index moves on a straight line passing through S, the stigma moves on another straight line also passing through S. If the index move on a straight line not passing through S, the stigma moves on a circle passing through S and conversely. If the index moves on a circle not passing through S, the stigma moves on another circle also not passing through S, and in an opposite direction of rotation.

If the whole group of stigmata be removed without disturbing their mutual relations, and so that the solitary index S no longer corresponds with the solitary stigma Z', a s. *homography* results (93), in which the relations of index and stigma may be deduced from those in a s. involution. These relations are expressed by the equation $sx \cdot zy = sa \cdot z'a'$, where (xy) , (aa') are s. points in the s. homography. When three s. points are given, the solitary index S, and solitary stigma Z', and double stigmata E, F can be determined by an elementary geometrical construction in all cases. These double stigmata possess important *angular properties* with respect to the indices and stigmata of the system (94). Let (aa') , (bb') , (cc') be any three s. points in a s. homography. Then if ABC, A'B'C' are straight lines (in which case the former contains S, and the latter Z'),

$$\tan AEA' = \tan BEB';$$

that is, *if two intersecting straight lines EA, EA' revolve about their point of intersection E, and cut two given straight lines AB, A'B', determining an index A upon one, and a stigma A' upon the other, so that the ordinate AA' is seen under an angle whose tangent is constant**, the s. points will form part of a s. homography, of which the given point E is one double stigma, the solitary index S lying upon the index line AB, and the solitary stigma Z' upon the stigma line A'B'. If O be the middle point of SZ' and F the other double stigma, FO = OE. If the two lines AB, A'B' coalesce, then the two points E, F, from which the ordinates AA', BB', CC' of three or more s. points, of which both indices and stigmata lie upon one straight line, are seen under an angle whose tangent is constant, are the double stigmata of the s. homography determined by these three s. points. This shows the nature of the points named in G. S. 171 †, and completes it, so far as "real rays" are concerned. It is further generalized hereafter. The

* It is customary, but erroneous, to say in such cases that the angles are equal and in the same direction. It is evident that one angle may be the supplement of another measured in the opposite direction.

† Charles's *Géométrie Supérieure* and *Sections Coniques* are referred to by the letters G. S. and S. C., followed by the number of the article.

two lines $AB, A'B'$ may be so placed that the $s.$ homography has only one double stigma O coinciding with the middle point of SZ' , and O will then be the centre of a circle of which $AB, A'B'$ are tangents; the ordinates of the $s.$ points in the $s.$ homography (each of which contains an index and a stigma) will then be tangents to the circle of which the centre (being a double stigma) will also contain an index and stigma. Hence is immediately obtained the relation in G. S. 664, and an explanation of the remark there made, that "*le centre du cercle tient lieu en quelque sorte d'une tangente.*"

The tangent of the angle subtended by an ordinate at the double stigma may be constantly zero, that is, the ordinates may converge to the double stigma. Then, *if through any point E two straight lines be drawn, and through any other point F rays be drawn intersecting the first line in a series of indices A, B, C , and the second in a series of stigmata A', B', C' respectively, the resulting $s.$ points $(aa'), (bb'), (cc')$ will form part of a $s.$ homography of which E, F are the double stigmata.* This shows the nature of the points of issue and intersection in the fundamental proposition of anharmonic ratios, G. S. 14, and completes it so far as "*real*" rays are concerned. It will be further generalized hereafter.

Hence $ABB'A'$ is a quadrilateral, of which E, F are the intersections of opposite sides. Consequently *if A, B' and A', B are opposite vertices of a quadrilateral, and E, F the intersections of the opposite sides $(ab'), (a'b), (ef)$ are $s.$ points in a $s.$ involution.* This completes G. S. 350, for "*real*" rays*.

The above considerations determine all the relations of $s.$ homography, but do not suffice to give an explanation of those "*imaginary*" rays which play so important a part in Chasles's geometry. The requisite theory is very simply obtained by considering the direction-points of two sets of (primary and secondary) $s.$ rays as indices and stigmata in a $s.$ homography or $s.$ involution (95). In particular it is shown that *if the $s.$ tangent of the $s.$ angle between two primary $s.$ rays is equal to that between the two corresponding secondary $s.$ rays, there are two pairs of parallel $s.$ rays, which are parallel to the asymptotes of a $s.$ circle*; and all the other relations for pencils of ordinary homographic rays are similarly generalized. In this case, if the direction-points of all the rays lie upon the same straight line passing through the origin, the $s.$ rays correspond to those usually termed "*real*;" and if any do not, the $s.$ rays belong to the class usually termed "*imaginary*," and were supposed not to have any geometrical existence.

When the two $s.$ points of issue of the two pencils are distinct, there will be generally two primary rays which are parallel to their secondaries;

* Some of these *results* of $s.$ involution and $s.$ homography have been previously obtained by Möbius (cited in the *Introductory Memoir*, 59 *t*); but his method and conception are perfectly different, and do not admit of the extension which the present propositions immediately receive, to include the case of "*imaginary*" rays, a subject never previously treated as strictly geometrical.

and if the two s. points of issue coincide (96), these parallel rays will become double rays. Both are found by finding the double points of the homography of direction-points. When the double rays are s. perpendicular*, the sum or difference of the s. tangents of the two s. angles made by the primary and its corresponding secondary with the double rays will be zero. If the direction-points form a s. involution, the s. rays form an involution, and there will be always two s. rays s. perpendicular and easily constructed.

The stigmata of the s. points in which the primary and secondary s. rays of two pencils, making the s. tangent of the s. angle between such rays constant, intersect two given s. lines, are indices and stigmata respectively of s. points in a s. homography, of which the stigma of the point of issue is a double stigma (97). Whence it follows that, in any s. homography, if the indices be considered as the stigmata of s. points on a known s. line, and the stigmata as the stigmata of s. points on some other known s. line, and a second index be given to one of the double stigmata so as to form a s. point which shall lie upon neither of those given s. lines, and pairs of s. rays be drawn from the s. point thus formed to pairs of corresponding s. points in the two s. lines, the s. tangent of the s. angle between any two corresponding rays will be constant.

Supposing each primary s. ray to coincide with its secondary, and OI to be the axis of reference (98), the stigma of the s. point of issue of a pencil of s. rays, and the stigmata of the intersection of these s. rays with a given s. line, form the indices, and the extremity I of the axis of reference OI, and the direction-points of the same s. rays, respectively form the stigmata of a series of s. points in a s. homography. Whence it follows that the anharmonic ratio of any four stigmata of intersection is the same as that of the four corresponding direction-points. This completely generalizes the fundamental proposition of anharmonic ratios, including the case where all the lines are "imaginary," and can be made the starting-point of a series of propositions precisely analogous to those in G. S., but with a much wider signification. Thus the property of the complete quadrilateral shows, on being generalized, that if E, A, B, C, D, E', A', B' be any eight points upon a plane, and two other points C', D' be assumed so that the triangles E'A'C', E'B'D' shall be directly similar to the triangles EAC, EBD respectively, and then two additional points F', F be determined so that the triangles F'A'B', F'C'D' shall be directly similar to the triangles FAB, FCD, the point F will be constant, if the first five points E, A, B, C, D remain unchanged, whatever the next three, E', A', B', may be, and will lie so that if SM be drawn bisecting the angles ASD, BSC and in length a mean proportional between the lengths of SA, SD, and also of SB, SC, then SM will also bisect the angle ESF, and be in length a mean proportional between SE and SF.

* If A and B are the direction-points of two s. lines, the latter are stigmatically perpendicular when $oa.ob=1$, and the direction-points are then harmonically situate with respect to I, I', where $OI'=IO$.

When pairs of separate s. rays cutting one s. line are considered, it is shown (99) that *if from any point pairs of ordinary rays be drawn which are the two limbs of a constant angle cutting any straight line in points forming the indices and stigmata of s. points in a s. homography of which the point of issue is a double stigma, and we furnish all the points lying on the given straight line, and the point of issue, with Cartesian indices, and hence consider all the s. rays as Cartesian, they will form part of an homography of s. rays, of which the double rays drawn from the Cartesian point of issue will be two non-Cartesian s. lines parallel to the asymptotes of a s. circle, and therefore independent of the magnitude of the constant angle.* This furnishes a complete geometrical explanation of G. S. 171, 172, 181, 651, S. C. 293, &c.

Generally the whole of the propositions in G. S. respecting the homography of ordinary points and rays may be transferred to the homography of the stigmata of s. points and of s. rays, forming a series of propositions of much greater generality, in which all the s. points and s. rays will be perfectly real, and can be constructed by means of elementary geometry.

The last division of this memoir is devoted to s. conics, which embrace the ordinary Cartesian conics as particular cases, together with all their so-called "imaginary" points and lines. The general equation is assumed (100) in the form

$$\alpha \cdot ox^2 + 2\alpha\beta \cdot ox \cdot oy + \alpha\gamma \cdot oy^2 + 2\alpha\delta \cdot ox + 2\alpha\epsilon \cdot oy + \alpha\zeta = 0; \quad (a)$$

and it is shown that if $\alpha\beta^2 = \alpha \cdot \alpha\gamma$ the equation may represent a single s. line, or two parallel s. lines, or a s. *acentric*, which, by changing the origin and index, may be made to depend on the equation $x'y^2 = 4s'o' \cdot o'x'$; but if $\alpha\beta^2$ is not $= \alpha \cdot \alpha\gamma$, the equation may represent two intersecting s. lines, or a s. *centric*—that is, a s. curve which when cut by any s. line passing through a given s. point (the s. *centre*) has the index and stigma of that s. point in the middle of the lines joining the two indices and two stigmata of the s. points of intersection respectively. It is also shown that if the origin and index are changed, the s. centric may be represented by the equations

$$\frac{o'x'^2}{o'e^2} + \frac{x'y^2}{o'g^2} = 1, \text{ or } o'x'' \cdot x''y = o'h^2, \quad (b)$$

in which case the s. lines $o'x' = 0$, $x'y = 0$ are indeterminate, but form *pairs of s. rays in involution*, which are all *conjugate diametrics**, with the exception of the double rays $o'x'' = 0$, $x''y = 0$, which are *asymptotes*. In the particular case that $\alpha\zeta = 0$, the general equation can, by changing origin and index, be made to depend on the equation $o'x'^2 - x'y^2 = o'e^2$. Such curves are called s. *cyclics*. In the cyclic proper, the values of $\alpha\beta$, $\alpha\gamma$ may be any whatever; in the s. *circle* $\alpha\beta + \alpha\gamma = 0$, in which case *all the conjugate diametrics are s. perpendicular, and the direction-points of the asy*

* A *diametric* is a s. line passing through the s. centre and intersecting the s. curve in two s. points. A *diameter* is the straight line joining the two stigmata of these last s. points. A *radius* is half a diameter. These distinctions are important.

ptotes are I, I'. But if both $o\alpha$ and $o\chi$ become $=0$, the s. cyclic is a s. homography or s. involution.

The general homographic properties of all s. conics are then deduced (101). First, *a s. conic is the locus of the s. points of intersection of the primary and secondary s. rays of two pencils of homographic s. rays with different s. points of issue, and passes through these points*, which, with slight modifications, holds when a band of parallel s. lines is substituted for either or both of the pencils. Secondly, *the anharmonic ratio of four s. chords in a s. conic drawn from a variable s. point to four fixed s. points, bears a constant ratio to the anharmonic ratio of the stigmata of those four fixed s. points; and this constant ratio is one of equality when the s. conic is a s. circle*. In order to establish the latter part of this proposition, it is shown that in a s. circle, the s. tangent of the s. angle formed by any two s. rays drawn from a variable s. point to any two fixed s. points in the s. circle is constant (which is a generalization of Euc. iii. 21), whence is deduced the condition that four s. points should lie upon a s. circle. The classification of s. centrics is founded (102) on the first of the equations (b). After showing how the two stigmata can be generally found from the index, the name of the s. centric is made to depend on the name of the locus of the stigma when the index describes the principal diameter in whole or in part. This locus is called the *characteristic curve*. According as $o'e^2 \div o'g^2$ is a negative scalar, -1 , a positive scalar, $+1$, or any clinant, the s. centric is a s. *ellipse, circle, hyperbola, equiradial* (equilateral hyperbola), or *hyperellipse* (for which the characteristic is a confocal ellipse and hyperbola). The mode of finding the two indices corresponding to each stigma is then given (103), and the circumstances under which a s. line and a s. centric may have common points investigated (104). The mode of determining s. points of intersection, when there is no intersection in Cartesian geometry ("imaginary" intersections) is illustrated, and shown to lead to the same results as the homographic method.

The *radical axis* (105) is shown to be the *Cartesian portion of a s. line which is the common s. chord of a series of s. circles, so situated that the extremities of their principal diameters are the indices and stigmata of a s. involution*, of which the "real" and "imaginary" circles of Chasles are two particular cases.

The properties of *asymptotes*, and the nature of those in a s. ellipse, are then considered (106), and their geometrical properties examined. *Conjugate Diametrics* and their relations to the asymptotes are next investigated, (107). In especial it is shown that if $(u'e')$, $(v'g')$ are two of the s. points in which two conjugate diametrics intersect a s. centric, and if the s. line drawn from $(u'e')$ s. perpendicular to the diametric connecting $(o'o')$ and $(v'g')$ meet the latter in a s. point whose stigma is D, we shall have

$$o'u' \cdot u'e' + o'v' \cdot v'g' = 0, \quad \dots \dots \dots (c)$$

$$o'u'^2 + o'v'^2 = o'e'^2, \quad u'e'^2 + v'g'^2 = o'g'^2, \quad \dots \dots \dots (d)$$

$$\text{whence} \quad o'e'^2 + o'g'^2 = o'e'^2 + o'g'^2, \quad \dots \dots \dots (e)$$

$$\text{and} \quad de' \cdot (o'v' - v'g') = \pm o'e' \cdot o'g' \quad \dots \dots \dots (f)$$

From these relations, which are much more general than any hitherto enunciated, the ordinary relations in Cartesian geometry respecting the lengths of conjugate diameters and the areas they contain are readily deduced.

By referring s. centrics to conjugate diametrics as new coordinate axes (108), a simple general method for constructing the intersections of a s. line with a s. centric, and for drawing two s. tangents (109) from any s. point (except the s. centre) is obtained. This is illustrated by actual geometrical constructions corresponding to "imaginary" cases in Cartesian geometry. It is also shown how the *chord of contact* or *polar* may have a Cartesian portion when both the s. points of contact are non-Cartesian and hence "imaginary."

It is now possible (110) to demonstrate the fundamental anharmonic properties of tangents and asymptotes to a s. centric. *If through four s. points in a s. centric, four tangents be drawn, intersecting any fifth tangent, and also four s. chords meeting in any fifth s. point in the s. centric, the anharmonic ratio of the four stigmata of the s. points of intersection of the four tangents with the fifth will be equal to the anharmonic ratio of the four s. chords* (which generalizes S. C. 2, 5, the demonstration being here direct and therefore widely differing from Chasles's), whence, *if two fixed tangents are cut by a moveable tangent the stigmata of its s. points of intersection with them will be the indices and stigmata of s. points in a s. homography*. It is now possible to generalize the whole of Chasles's S. C., and make the theories applicable, *mutatis mutandis*, to s. conics.

The s. foci of a s. centric (111) are defined as the s. points of intersection of any two tangents which are parallel to the asymptotes of a s. circle. Hence it is shown that there are four s. foci to a s. centric; so that if $o's^2 = o'e^2 + o'g^2 = o'z^2$, the two principal s. foci are (*ss*) (*zz*), and the two transverse s. foci are (*o's*), (*o'z*). When the s. centric is Cartesian, the two principal s. foci are those usually recognized, and the only two recognized by Chasles, S. C. 275, 294. The other two transverse s. foci have, however, been recognized as "imaginary" foci by Plücker (*System der a. Geometrie*, p. 106, l. 6), and by Salmon (*Conics*, 4th ed. p. 242).

Besides the usual properties of the focal stigmata in Cartesian curves, the following entirely new correlative properties of the principal and transverse s. foci are demonstrated. *Let there be two conjugate diametrics, and through any s. point in one draw a s. line parallel to the other, and let the stigmata of the s. points of intersection of the conjugate diametrics and the chord, with the s. centric, be respectively E', F'; G', H'; Y₁, Y₂. From E', F' draw straight lines PARALLEL to the bisectors of the angles Y₁SY₂, Y₁ZY₂ respectively, forming two sides of a triangle of which E'F' is the base. Then the lengths of these sides will be mean proportionals between the lengths of SY₁, SY₂, and ZY₁, ZY₂ respectively.* This is a generalization of the usual property that the sum or difference of the

focal distances is the axis major in the ellipse or hyperbola. *Preserving the same notation, take X' the middle point of Y_1Y_2 , set off $X'X_3=X_4X'$ $=O'S$, and make $X_3Y''=Y_2X_3$, and $X_4Y'''=Y_2X_4$. From the extremities G',H' of the diameter CONJUGATE to that used in the last proposition, draw straight lines PERPENDICULAR to the bisectors of the angles $Y''SY_2, Y'''ZY_2$ respectively, forming the two sides of a triangle of which $G'H'$ is the base. Then the lengths of these sides will be mean proportionals between the lengths of SY'', SY_2 and ZY''', ZY_2 respectively.* This is the correlative property of the transverse s. foci, which has no Cartesian analogue. The deduction and expression of these properties is extremely simple, and they are illustrated by a geometrical construction. Finally, *the s. tangent of the s. angle which the s. rays drawn from any s. point in a s. centric to the two principal or the two transverse s. foci make with the s. normal at the s. point are equal and opposite*, which is a generalization, for both pairs of s. foci, of the property from which the ordinary foci derived their name.

The following problems are then solved (112). First. *Given two conjugate radii $O'E', O'G'$ to find the principal and transverse axes of the Cartesian centric.* The focal stigmata S, Z are found from (e) thus: from E' draw $E'M, E'M'$ perpendicular to $O'G'$, and equal to it in length; join $O'M, O'M'$, bisect the angle $MO'M'$ by $O'S, O'Z$, which are in length mean proportionals between $O'M, O'M'$. Then there are three solutions, giving an ellipse or two hyperbolas, according as both or only one of the conjugate radii meet the characteristic curve of the Cartesian centric. Second. *Given any three s. points representing the s. centre, and the s. points of intersection of two conjugate diametrics with the s. centric, to find its principal and transverse axes.* Third. *Given three s. points, to find the axis of a s. circle passing through them.* Fourth. *Given any five s. points, to determine whether they lie on a s. centric; and if so, to find its principal and transverse s. axes.*

Equation (e) is then applied (113) to obtain a more general conception of confocal s. centrics, and Carnot's theory of transversals (as generalized in the *Introductory Memoir*) is applied to finding a new expression for the s. radius of curvature, applicable for any s. point of contact. The ordinary expression gives length and not direction, and applies only to Cartesian points. Similar investigations are very briefly given for s. *acentrics* (115-118).

In these memoirs on Plane Stigmatics, the writer has endeavoured to confine himself strictly to such a development of his theory as would suffice to establish, with geometrical severity, the following results, which he believes to be entirely new, and of great scientific importance to mathematics.

First. *The problem of the geometrical signification of imaginaries in plane geometry is completely solved; so that the terms "real" and "imaginary" are no longer required, and an unbroken agreement exists between plane geometry and ordinary commutative algebra.*

Imaginaries arise whenever an algebra is used which is more general than the geometry to which it is referred. Ordinary commutative algebra is an algebra of clinants, which corresponds to a geometry where not only the relative lengths, but the relative angular positions of straight lines are regarded. Now the geometry hitherto associated with it has been scalar, that is, has considered relative lengths, but only identity or opposition of direction. Thus in Cartesian geometry the abscissæ and ordinates were necessarily supposed to be parallel to given lines; and in homographic geometry, where the ordinates were not taken into account, their extremities were supposed to lie on one or two given lines. But the algebra employed in either case took no notice of these restrictions, and hence led to results which could not be interpreted—that is, “imaginaries.” The solution of the problem of imaginaries has, therefore, consisted in that stigmatic conception which removes these restrictions, freely introduces relative angular position, and makes the geometry coextensive with the algebra. In the course of these memoirs every fundamental case in which imaginaries occur has been separately examined and shown to be fully explained by this conception.

Second. *The coordinate geometry of Descartes, and the homographic geometry of Chasles, are only particular cases of the writer's stigmatic geometry, identical in nature, and expressible by identical equations;* for both Cartesian and homographic geometry consist in the relation of index to stigma with various restrictions, which may or may not be regarded in stigmatic geometry.

Third. *The general theories of plane curves, as treated by ordinary co-ordinate geometry, by the systems of coordination introduced by Plücker, or the modes of investigation more recently developed by Chasles, and all the theories derivable from these, hold, IN THEIR INTEGRITY, for stigmatic curves ALONE.*

For in all these theories clinant algebra is associated with scalar geometry, for which the stigmatic conception enables us to substitute that clinant geometry which perfectly agrees with the algebra employed; and the fundamental propositions of these theories have been extended accordingly in the course of these memoirs.

The instrument by which the writer has been enabled to bring these investigations to a successful issue, has proved so serviceable and manageable, embracing old and new properties in a single equation, often rendering a general investigation more easy to conduct than the former special researches, and keeping the geometrical operations indicated clearly before the mind, that he would add as a subsidiary result of his theory:—

Fourth. *A calculus and a notation have been invented and developed, which enable magnitude and direction upon a plane to be expressed by a single symbol, obeying the laws of ordinary algebra, closely resembling in appearance (not in theory) the notation employed by Chasles to represent magnitude and direction upon a straight line, and as easy of manipulation.*